The String Equations

for Solid and Wound Musical Instrument Strings

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1. Introduction

The description of a vibrating string is a well-known problem in physics. It has been extensively described in countless textbooks, and in many articles on the Internet (see, e.g., Refs [1-4]). However, a detailed description of the behavior of a wound string (a string consisting of a core wire with additional windings to increase the string mass) is not as easy to find. This article derives the string equations for both solid and wound strings. For completeness, the derivation of string equation itself is also presented.

2. The Solid String

Consider a stretched string with uniform diameter d_c. We assume that the string obeys the 'ideal string model', i.e.:

- the string has uniform mass density
- bending stiffness is negligible
- effects of gravity are negligible
- there is no dissipation
- the string moves only in the transverse direction.

Although stringent, it has been demonstrated that this ideal string model efficiently describes the behavior of most musical instrument strings in practice.

Fig. 1 shows a short segment of a solid string, which is stretched along the horizontal (x-) axis. The shape of the string is described by the function $z(x,t)$. The vertical position of the string segment runs from z *to* $z+\Delta z$. The tangent to the string segment makes an angle α_1 with the horizontal at position x, and an angle α_2 at position $x+\Delta x$. The internal string tension (force) may vary over the string, and amounts to T_1 and T_2 at positions x and $x+\Delta x$ respectively.

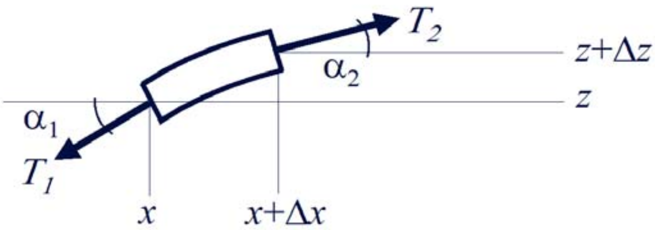

Figure 1*: string segment in ideal string model*

The net vertical force F_z acting on the string segment amounts to:

$$F_z = T_2 \sin(\alpha_2) - T_1 \sin(\alpha_1) \qquad [1]$$

The ideal string model requires that the net horizontal force acting on the string segment be zero, so that

$$T_1 \cos(\alpha_1) = T_2 \cos(\alpha_2) = T \qquad [2]$$

Combining these two equations, we find that

$$F_z = T\left(\tan(\alpha_2) - \tan(\alpha_1)\right) \qquad [3]$$

From the geometry it follows that

$$\tan(\alpha_1) = \left(\frac{\partial z}{\partial x}\right)_x \qquad \tan(\alpha_2) = \left(\frac{\partial z}{\partial x}\right)_{x+\Delta x} \qquad [4]$$

If the specific mass of the string material equals ρ_c, the mass dm of the string segment amounts to:

$$dm = \Delta x \frac{\pi d_c^2}{4} \rho_c \qquad [5]$$

Applying Newton's second law $F = ma$ on the string segment, we get for the vertical motion, using Eqs. [3], [4] and [5]:

$$T\left(\left(\frac{\partial z}{\partial x}\right)_x - \left(\frac{\partial z}{\partial x}\right)_{x+\Delta x}\right) = \Delta x \frac{\pi d_c^2}{4} \rho_c \frac{\partial^2 z}{\partial t^2} \qquad [6]$$

In the limit Δx→0, this expression takes the from

$$\frac{\partial^2 z}{\partial x^2} = \frac{\pi d_c^2 \rho_c}{4 T_x} \frac{\partial^2 z}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} \,. \qquad [7]$$

This is the well-known one-dimensional wave equation, with general solution

$$z(x,t) = A(x - ct) + B(x + ct) \qquad [8]$$

where

$$c = \frac{2}{d_c} \sqrt{\frac{T_x}{\pi \rho_c}} \,. \qquad [9]$$

Here $A(x)$ and $B(x)$ are arbitrary, real functions. The two terms describe traveling waves in the string: any point on the wave form given by a A or B will move with velocity c in either the forward or backwards direction: forwards for A and backwards for B.

3. The String Equation for the Solid String

The internal string tension follows from Eq. [9]:

$$T = \frac{c^2 d_c^2 \pi \rho_c}{4}. \qquad [10]$$

In the case of a string with finite length L, we must apply the boundary conditions $z(0)=0$ and $z(L)=0$. This implies that the largest wavelength that can be accommodated by the string corresponds to $\lambda=2L$. The length L and the velocity c determine the frequency by $f = c/\lambda$, so that Eq. [10] then gets the form

$$T = L^2 f_0{}^2 d_c^2 \pi \rho_c. \qquad [11]$$

Note that in this expression the frequency f_0 corresponds to the lowest frequency the string can support - the fundamental frequency - because the wavelength λ has been substituted by $2L$. Eq. [11] is known as the *string equation* for the solid string.

3. The String Equation for the Wound String

It will be clear from the analysis so far, that Eq. [11] is valid for both solid strings and wound strings, as long as the ideal string model is not violated. In fact, because wound strings are deliberately designed for low stiffness, they are usually well described by the ideal string model. However, the additional mass of the winding wire must be taken into account appropriately.

We will now derive the equivalent of Eq. [11] for a wound string, by evaluating an appropriate substitute for the mass density ρ_c. Consider a wound string of length L, core diameter d_c , winding diameter d_w and internal tension T, as shown in Fig. 2.

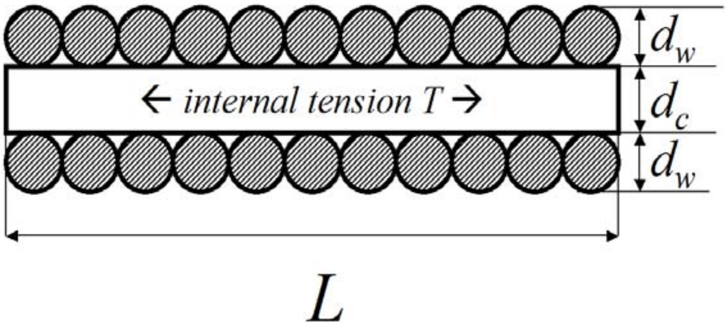

***Figure 2**: cross section of wound string*

For simplicity, the windings are described as a collection of toruses rather than a spiral. The number of windings N will amount to:

$$N = \frac{L}{d_w} \qquad [12]$$

The tension will only act in the core, not in the windings. Within the ideal string assumptions, this has no effect on the string dynamics, as long as the windings remain fixed to the core.

The mass of the core, m_c, amounts to:

$$m_c = \frac{\pi d_c^2}{4} L \rho_c . \qquad [13]$$

Similarly, the mass of the windings, m_w, amounts to

$$m_w = \frac{\pi d_w^2}{4} L_w \rho_w \qquad [14]$$

where L_w is the total length of the winding wire, and ρ_w the specific mass of the winding material. It is readily found that

$$L_w = \frac{L}{d_w} \pi (d_c + d_w) \qquad [15]$$

After substitution of Eq. [15] into Eq. [14] and adding Equation [13], we find the expression for the total mass m of the wound string:

$$m = \frac{\rho_c L \pi}{4} \left[d_c^2 + \frac{\rho_w}{\rho_c} \pi d_w (d_c + d_w) \right] \qquad [16]$$

We define the composite mass density ρ of the wound string as the mass m divided by the volume V, where

$$V = \frac{\pi L}{4} (d_c + 2 d_w)^2 \qquad [17]$$

i.e., we consider the wound string as a cylinder with length L and diameter $d_c + 2d_w$. This is a logical choice, because the diameter $d_c + 2d_w$ is also specified by commercial string manufacturers for the ‘gauge’ of wound strings. The composite mass density ρ then becomes, after dividing Eq. [16] by Eq. [7]

$$\rho = \rho_c \frac{d_c^2 + \frac{\rho_w}{\rho_c}\pi d_w(d_c + d_w)}{(d_c + 2d_w)^2} \qquad [18]$$

After replacing d_c by the wound string diameter $d_c + 2d_w$ in Eq. [11], and using Eq. [18] to replace the factor ρ_c, we find for the total tension acting on the wound string:

$$T_w = \pi f^2 L^2 d_c^2 \rho_c \left(1 + \frac{\rho_w}{\rho_c}\pi \frac{d_w(d_c + d_w)}{d_c^2}\right) \qquad [19]$$

This is an interesting expression, because Eq. [19] is identical to the expression for the tension in a solid string with diameter d_c, as written in Eq. [11], with a correction factor equal to

$$1 + \frac{\rho_w}{\rho_c}\pi \frac{d_w(d_c + d_w)}{d_c^2}.$$

If we assume that $\rho_w = \rho_c$ (identical core and winding materials) and $d_c = d_w$ (identical core and winding diameter) we find that this correction factor equals to $1+2\pi = 7.28\ldots$, i.e. most significant. This is because in Eq. [19] we still consider the diameter d_c as the diameter of the wound string, which is in fact a factor of three less than the actual external diameter of the wound string. Hence, the correction factor needs to take this discrepancy into account.

It is more consistent to rewrite Eq. [19] in the following form:

$$T_w = \pi f^2 L^2 (d_c + 2d_w)^2 \rho_c \frac{\left(d_c^2 + \frac{\rho_w}{\rho_c}\pi d_w(d_c + d_w)\right)}{(d_c + 2d_w)^2}, \qquad [20]$$

which can be interpreted as the expression for the tension of a solid string, with mass density ρ_c and diameter $d_c + 2d_w$ (i.e., the *external* diameter of the wound string), with a ‘winding correction factor’ equal to

$$\frac{d_c^2 + \frac{\rho_w}{\rho_c}\pi d_w(d_c + d_w)}{(d_c + 2d_w)^2} \qquad [21]$$

With the same assumption as before (identical string wire for core and winding) the correction factor now becomes $(1+2\pi)/9 = 0.81$. This factor can be interpreted as a correction to the specific mass of a solid string with diameter $d_c + 2d_w$, in order to compensate for the ‘air between the windings’.

4. The String Equations in Practice

String manufacturers usually do not provide the diameters of core and winding wires, so it is not always simple to determine the correction factor in Eq. [21]. The table below lists some typical diameters of core and winding wires of commercial guitar strings, along with the resulting correction factors:

string gauge	d_{core}	d_w	correction factors	
[inch]	[m]	[m]	bronze	phosphor bronze
0.024	3.30E-04	1.35E-04	0.880	0.918
0.032	3.81E-04	2.10E-04	0.867	0.909
0.042	4.06E-04	3.22E-04	0.854	0.900

Table 1*: winding correction factors for some typical commercial wound strings*

The correction factors were calculated for the case of a steel core wire (mass density = 7874 kg/m^3) with bronze or phosphor bronze windings (mass density 8300 and 8850 kg/m^3 respectively). From the table it is seen that a correction factor of approximately 0.9 is a fair estimate for a typical wound string.

Some string manufacturers list the tension for each string on their packages, so that we can verify the validity of our models by comparing the specified tension for a commercial set of strings with the tension predicted by Eqs. [11] and [20]. The results are shown in table 2. A fixed correction factor of 0.9 was taken for all wound strings.

string	frequency	diameter	tension (specified)	tension (calculated)	error
	[Hz]	[inch]	[kg]	[kg]	%
E	82	0.053	11.79	11.60	-1.6
A	110	0.042	13.56	13.11	-3.3
D	147	0.032	13.83	13.59	-1.7
G	196	0.024	13.70	13.59	-0.8
B	247	0.016	10.57	10.66	0.8
E	330	0.012	10.57	10.70	1.3

Table 2*: specified and calculated string tensions for a commercial set of guitar strings (D'Addario EXP16). The assumed mass density of the core wire is 7874 kg/m3, and a winding correction factor of 0.9 was used for all wound strings (E, A, D and G), string length = 25.5 inch (0.6477 m), which is standard for most steel string guitars. The gravitational acceleration was taken as 9.81 m/s^2 to convert N into kg.*

It is seen that with the (estimated) correction factor of 0.9 our model is accurate within a few percent, which is more than sufficient for most practical purposes. More accurate results can only be obtained after precise measurement of the core and winding wires, verified values for the mass densities of the materials used, and possibly also a correction factor for the (typically) hexagonal shape of the core wire.

The high accuracy of our model allows the use of simple ‘rules of thumb’, derived from Eqs. [11] and [20], such as the observation that the diameter of a string (wound or solid) must be inversely proportional to the fundamental frequency it produces, in order to maintain the same string tension. This fact allows the quick calculation of required string diameters for alternate tunings. For example, an E-string (82 Hz) of 0.053 inch, tuned down two steps to a D (74 Hz) will have a reduced string tension. A replacement string with a diameter of 82/74*0.053 = 0.059 inch will have the same string tension when tuned to 74 Hz as the original string at 82 Hz. This simple rule is valid for both solid and wound strings (again, assuming identical winding correction factors), and makes it easy to define custom sets of strings for alternate tunings, with even string tension.

5. Conclusions

Within the assumptions of the ideal string model, we derived the string equations for solid and wound strings. These equations relate string tension, diameter, length, mass density and fundamental frequency of the strings for musical instruments. Within the used model, wound strings have the same dynamics as solid strings, but require a winding correction factor to account for the geometry of the winding wires. This factor is depending on the used materials and the specific geometrical properties of the wound string. However, it was demonstrated that a fixed winding correction factor of 0.9 produces results that are accurate within a few percent.

6. References

[1] http://www.math.duke.edu/education/ccp/materials/engin/wave/wave1.html
[2] http://hyperphysics.phy-astr.gsu.edu/hbase/waves/waveq.html
[3] http://ccrma.stanford.edu/~jos/pasp/Non_Stiff_String.html
[4] http://www-solar.mcs.st-and.ac.uk/~alan/MT2003/PDE/node11.html